Extension of the Black-Litterman model: an empirical experiment on a multi-asset portfolio and comparison to the original model

**Gleb Ilyasov**

Supervisor: Professor Kim Oosterlinck

Assessor: Professor Hugues Pirotte

A master thesis submitted for the degree of Business Engineering, major in Advanced Management (Finance)

Academic year 2018-2019

**Abstract**

**Acknowledgements** I am grateful to …

**Keywords** Minimum Relative Entropy, Entropy Pooling, Fully Flexible Probabilities, Black-Litterman Model, Mean-Variance, Modern Portfolio Theory, Conditioning, Portfolio Optimisation, Global Minimum Variance, Asset Allocation, Stress Testing, Subjective Views.

**Contents**

1 Introduction

2 Literature Review

3 Methodology

4 Experimental Set-Up

5 Results

6 Conclusion

**List of Figures**

**List of Tables**

**1 Introduction**

Asset allocation has always been a matter of interest, already in the fourth century, Rabbi Issac bar Aha proposed the following rule: “One should always divide his wealth into three parts: a third in land, a third in merchandise, and a third ready to hand.”1

The Modern Portfolio Theory (Markowitz, 1952) shows considerable advances and is a cornerstone in modern investment theory. Markowitz argues that investors should rely on the first two moments of the return distribution2 to optimally allocate assets. However, three well-documented problems with the mean-variance optimisation exist: highly concentrated portfolios, input-sensitivity, and estimation error maximization. These related issues also lead to poor out-of-sample performance as shown by DeMiguel, Garlappi, Uppal, 2007). These drawbacks are the most likely the reasons that more practitioners do not use the Markowitz paradigm.4

To tackle some of the mean-variance optimization approach’s limitations, Fischer Black and Robert Litterman have elaborated a pathbreaking technique. “The Black-Litterman model uses a Bayesian approach to combine the subjective of an investor regarding the expected returns of one or more assets with the market equilibrium vector of expected returns (prior distribution) to form a new, mixed estimate of expected returns.”3

A plethora of tweaked Black-Litterman models have been published over the years such us Martellini & Ziemann (2007), Meucci (2010), Cheung (2013), Harris (2017) and many others. Black Litterman Model and its alternatives are characterized by their need for a normally distributed set of returns and views as well as its Bayesian nature. This constitutes a major imperfection as financial markets are well-known for displaying fat tails, Financial Risk Manager Handbook (Jorion, 2007). Moreover, “the combination of a negative skewness and a fat tail has the greatest impact on the optimal asset allocation weights” (Xiong & Idzorek, 2011).

Meucci (2005) extends the Black-Litterman methodology to generic non-normal market distributions and non-normal views. In his model, views are directly expressed on the market realizations, contrarily to the Black-Litterman model, where views are expressed on market parameters. Meucci’s Copula-Opinion Pooling model is to use opinion pooling techniques to combine the market implied distribution with the portfolio manager’s views which can be specified individually.

Finally, Entropy Pooling (Meucci, 2008) is a model based on entropy minimisation, a concept used in physics and statistics yet with limited applications in finance. In this new approach, Meucci generalizes his previous works and related techniques. A reference market model (called “prior”) and general views are the only inputs needed to compute the posterior distribution. The views are interpreted as statements that modify the probabilities associated with each historical scenario of the prior distribution, they are expressed as constraints on the original market probability distribution, itself remaining unaltered. The posterior distribution should contain the least possible amount of spurious, fallacious structure; this is obtained by minimizing the entropy relative to the prior distribution. A portfolio manager applying the Entropy Pooling model can express views not only on expectations of asset performance but rather on any property of the reference model.

In this paper we use Fully Flexible Probabilities Conditioning Framework (Denis & Scaillet, 2017) after the initiating work of Meucci (2010)[[1]](#footnote-1). The algorithm developed in the cited paper allows one to modify the “weight” of a range of historical scenarios or Monte Carlo simulations according to his subjective views on the market. The posterior distribution is created from the conditioned scenarios and is then used for global minimum variance optimisation resulting in an asset allocation following the practitioner’s subjective views.

Two types of conditioning were be expressed, namely “time” and “state” conditioning. By the means of a rolling window or an exponential smoothing, the time conditioning allows the manager to emphasise historical scenarios according to the moment they occurred. The second conditioning is based on any macro-economic indicator chosen by the portfolio manager for which he/she wants to accord importance to. The algorithm is capable of exerting both time and state conditioning through minimum relative entropy. The combination allows to obtain an interesting procedure as it will give more weight to both recent observations and particular market conditions.

In the research paper of Denis Marvin and Scaillet Olivier (2017) the data used was composed of SPDR ETFs, resulting in a stocks only portfolio. In this paper, the portfolio will be constituted of multiple asset types, including alternative investments. We expect the results to be of interest due to the increasing trend of investment managers to include different types of assets in their portfolio. According to Bessler, Opfer and Wolff (2017), Black-Litterman optimized portfolios significantly outperform naïve-diversified portfolios, and consistently perform better than mean-variance, Bayes-Stein, and minimum-variance strategies even after controlling for transaction costs. There is therefore an interest in comparing the Black-Litterman optimized portfolios to the algorithm based on Entropy Pooling (Meucci, 2010) studied in this paper.

Our illustrative empirical experiments consist in a global minimum variance optimisation based on 6 years of daily data. The multi-asset portfolio is composed of: Fixed Income, Equities, Real Estate (REITs), Commodities, Cash and cash equivalents, Foreign Exchange (Currencies) and Cryptocurrencies.[[2]](#footnote-2)

Our objective with this paper is to measure if applying this tool to asset allocation in a multi-asset portfolio can help reduce the general risk of the said portfolio. This objective is assessed by answering the following questions:

1. Does the minimum relative entropy asset allocation allow to achieve the best portfolio performance in comparison to other strategies following the various performance measures?
2. What macro-economic indicators lead to the best result?
3. (Does the algorithm perform well with monthly rebalancing on daily data?)

The remainder of this paper is organized as follows. In Section 2 presents a brief summary of Markowitz’s Modern Portfolio Theory. We then explain the implementation of views using Black-Litterman model and mention their limitations. An explanation of the Entropy Pooling method will then be provided followed by a discussion on performance measures. In Section 3 we present our methodology starting with historical scenarios enhancement through Fully Flexible Probabilities and minimum relative entropy and then explain the two-step mean-variance framework we use for global minimum variance optimisation. In Section 4 the data set we utilize for empirical experiments is presented and an explanation on how the portfolios are backtested is provided. We present the obtained results in Section 5 and finally conclude in Section 6.

1. **Literature Review**
   1. **Mean-Variance Framework**

The theory of portfolio analysis aims to find the sets of assets that appear to be efficient in a risk-return space. The various mix of assets that exhibit minimum risk for a given return or, alternatively, maximum return for a given level of risk are so-called efficient portfolios. The modern portfolio theory pioneered by Markowitz (1952) (1959), also known as the mean-variance approach, uses the mean and variance of the return distribution as measures of return and risk. The goal of this model is to minimize the risk while maximizing the portfolio’s expected return by optimally allocating financial securities in a portfolio while considering a given set of constraints.

The constrained maximisation problem may be expressed as follows:

A close up of a logo

Description automatically generated

where *w* ≡ [*w1, …, wN*]’is a column vector of portfolio weights, ∑ is a covariance matrix of asset returns, µ is a column vector of expected returns, λ is a measure of the investor’s risk aversion and *Ⅽ* is a set of constraints.

In this framework, it is assumed that the first two moments of the portfolio’s returns are sufficient to obtain an optimized portfolio allocation. The precision of the input parameters is therefore of prime importance. However, several drawbacks have been documented. First, the mean-variance framework is very sensitive to estimation errors of the mean and covariance matrix as demonstrated by Jobson and Korkie (1980), Michaud (1989), Chopra (1993). Indeed, small variations in the mean of a particular asset can force half of the assets from the portfolio (Best and Grauer, 1991) . Second, the usage of the two first moments often produce highly concentrated portfolios. In other words, some assets become overrepresented in the portfolio which leads to poor diversification (Yanushevsky and Yanushevsky, 2015). Last but not least, the performance of this model is not consistent across different samples. In some cases, the Mean-Variance framework performed worse than equally-weighted portfolios (Jorion, 1985). Additionally, DeMiguel, Garlappi and Uppal (2007) found that the higher idiosyncratic volatility in corner solutions in mean-variance portfolios leads to higher losses compared to equally allocating wealth across individual assets. Highly concentrated portfolios are in part due to high estimation errors in computing the mean returns (Danthine and Donaldson, 2005). Very large time series allow to estimate expected returns with more precision (Merton, 1980). DeMiguel et al. (2007) observe that for a mean-variance strategy to outperform the 1/N benchmark, the length of the estimation period would need to exceed 3000 months for a portfolio comprising 25 assets, and more than 6000 months for a portfolio with 50 assets in data sets of monthly returns. For a matter of comparison, these two first moments are typically estimated using periods of 60 to 120 months of data.

In a continuous research for tackling the previously mentioned issues, other asset allocation strategies have been developed. The Maximum Sharpe Ratio is one example, another is the global minimum variance (hereafter GMV) portfolio. This framework minimizes the volatility of a portfolio, assumingly measured by the variance, without putting any constraint on the expected returns. Note that the GMV portfolio is the optimisation method used for empirical experiments in this study as it is explained in Section 3.2.2.

Other attempts have also been made to develop more stable mean-variance portfolios in order to tackle with the effect of estimation errors in the estimates of expected returns. A widely spread belief is that estimation errors on risk are of less importance than errors in estimates of expected returns in the overall portfolio estimation risk (Ceria & Stubbs, 2006). Chopra et al. (1993) considers using a James-Stein estimator that shrinks the expected returns towards the average expected returns. Jorion (1985) proposes to shrink the expected return estimates towards the minimum variance portfolio. Bayesian estimation of means and covariances were also suggested by Klein and Bawa (1976), Frost and Savarino (1986) and Black Litterman (1990), with a main assumption that estimates have distributions too. Apart from building more robust optimizations, other authors introduced the implementation of subjective views in the optimization process (Goldfarb et al., 2003).

* 1. **Adding Subjective Views**

Specifying subjective views allows a portfolio manager to implement his expectations during a portfolio optimization. Investors are able to combine their unique views based on the information and signals they gathered about the performance of the portfolio’s securities with the market equilibrium. It allows them to obtain intuitive, diversified, and personalized portfolios. In that matter, the Black Litterman model has become one of the most commonly used asset allocation approaches in practice (Bevan and Winklemann, 1998; Bertsimas et al, 2012). Copula-Opinion Pooling (Meucci, 2006) and Entropy Polling (Meucci, 2008) are more recent models respective to the subjective views research path. The later will be posteriorly covered as the framework is a fundamental base of the algorithm used in this paper.

* + 1. **Black-Litterman**

The Black-Litterman (hereafter BL) model uses an equilibrium approach to estimate the expected returns of the portfolio’s securities. A Bayesian approach is then used to combine the investor’s subjective views with a prior market distribution to form a posterior distribution. The resulting new vector of returns (posterior distribution) is subsequently used for mean-variance optimisation, leading to less concentrated and more intuitive portfolios (Idzorek, 2004).

The main benefit in using BL over plain mean-variance framework is the more stable output. Using a “reverse optimisation”, BL obtain a distribution of returns from the equilibrium market portfolio. Subjective views can then be specified and seamlessly blended with the prior distribution. Confidence levels can also be set on the views, the higher the confidence the more the model tilts from a set of neutral weights towards the investor’s views (Black and Litterman, 1992).

In this paper, we follow He and Litterman (2002), Idzorek (2004) and Meucci (2010) to describe the Black-Litterman model. Some of the authors that have attempted to demystify the framework are Satchell and Scowcroft (2008), Walters (2008), Cheung (2010) among others.

* + - 1. **The Model**

**The market model**

We assume the market contains N different assets such as equities, bonds, currencies and others and that the returns of these assets are normally distributed. The market is therefore defined by a joint normally distributed time series of returns of N assets with *µ* the expected return and Σ the covariance matrix. The assumption is that the covariance is a known constant matrix.



**The prior**

In equilibrium, the same portfolio *weq* is hold by all investors as a whole. Indeed, if all investors hold the same view, the demand and the supply for assets are perfectly matched (Black, 1989). *µ* is set in terms of market equilibrium. Assuming all investors optimize their portfolio with a mean-variance framework without constraints, we have:

A picture containing object

Description automatically generated (3)

where λ represents the investor’s risk aversion level. By deriving the term between brackets with regard to **w** and equalling it to zero, we find the equilibrium portfolio weights

ŵ = (1/2λ)Σ-1ᴨ. By inverting this equation, we obtain the prior equilibrium expected returns also known as implied returns:

A close up of a logo

Description automatically generated (4)

This equation is also often expressed as follows: (Idzorek, 2004)

A screenshot of a cell phone

Description automatically generated (5)

with Π being the implied equilibrium return vector, λ the risk aversion coefficient, Σ the covariance matrix and *wmkt* the market capitalization weight of the assets.

Since *µ* in the market cannot be know with certainty, BL assumes it is a normally distributed random variable whose dispersion represents the possible estimation error. Leading to:

A picture containing object

Description automatically generated (6)

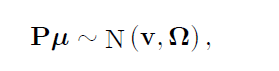
where ᴨ is the best guess for *µ* and τΣ the uncertainty on this guess. Assuming there is no estimation error, i.e. τ ≡ 0 in (6), the market model (2) becomes:

A picture containing object

Description automatically generated (7)

**The views**

A view in an opinion on the market that can potentially clash with the reference market model (2). For instance, the practitioner might have a prediction that the third security will be outperformed by the second asset class, the view would be *X3 – X2 ≤* 0. With the BL model portfolio managers have views on expectations, corresponding to statements on the parameter *µ* in the case of normal market (2). Assuming a *K x N* “pick” matrix **P** with K, the total number of views, the generic *k*-th row determines the relative weight of each expected return in the respective view. To associate uncertainty with the views, a normal model is used:

 (8)

where **v** and Ω are meta-parameters that respectively quantify views and uncertainty. Writing **v** as **v** = **P***µ* + **Z** where **Z** ~ N(0, Ω). **v** can now be expressed as a random variable **V** whose distribution conditioned on the realisations of *µ* is:

**V|**µ ~ N (**P**µ, Ω) (9)

where Ω should be computed by the practitioner. Different formulas are suggested by various authors; He and Litterman (2002) advocate Ω ≡ *diag*(*P*(τΣ)P’). Idzorek (2004) uses confidence in the views and Meucci (2010) uses

A close up of a logo

Description automatically generated (10)

where *c* ∈ (0, ∞) represents an overall level of confidence in the views with *c* = τ-1.

**The posterior**

Adapting from Satchell and Scowcroft (2000), the probability density function of (6) is

A screenshot of a cell phone

Description automatically generated (11)

taking into consideration (9), the condition probability density function of **v** is:

A picture containing sky

Description automatically generated (12)

Given **V**, Meucci applies Bayes rule to determine the posterior of ***µ***.

A close up of a logo

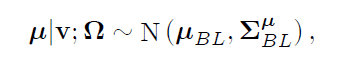
Description automatically generated (13)

The numerator of (13), being the joint pdf of *µ* and **V**, is equal to:

A close up of a device

Description automatically generated (14)

Meucci then obtains

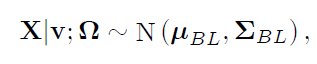
 (15)

where

A close up of a clock

Description automatically generated (16) & (17)

The reference model (2) can be written as **X** = *µ + (Z)* with **Z** ~ N(0, Ω) to obtain the distribution of the risk factors X. Therefore, the posterior market model is

 (18)

where *μBL* is expressed in (16) and *ΣBL* follows from (17), assuming that µ and **Z** are independent:

A picture containing object, clock

Description automatically generated (19)

Meucci (2010) has found an equivalent, more stable computational representation of the posterior parameters (16) and (19):

A picture containing object, sky

Description automatically generated

The reference model (2) is modified by incorporating the views (8) to obtain the normal posterior distribution (18) with parameters being (20) and (21).

The process of combining the sources of information can be visualized with Figure 1.[[3]](#footnote-3)

A close up of a logo

Description automatically generated

* + - 1. **Limitations**

Implicit assumptions of the BL approach are the following: (1) the returns of individual securities are normally distributed, (2) the expected return vector and the covariance matrix are constant over time, and (3) no differentiation is made between positive and negative deviations from the mean (Harris, Stoja and Tan, 2017). These assumptions remain questionable.

First, empirical evidence suggests that the distribution of returns are not normally distributed and display fat tails (Peiro, 1999; Ang and Chen, 2002; Jorion, 2007). Second, following Bollerslev et al. (1988), investors have time-varying conditional expectations of returns. Furthermore, most financial time series have time-varying variances and covariances (Andersen et al., 2010). Finally, Scott and Horvath (1980) claim that investors have asymmetric behaviours towards downside and upside risks. It also links to how should risk be measured, discussed later in Section 2.3.

* + 1. **Fully Flexible Views: Entropy Pooling**

Entropy minimisation is a largely used in physics and statistics, but less so in finance. Some authors suggest its use in the field such as Avellaneda (1998); D’Amico et al. (2002); Cont and Tankov (2006) and Pezier (2007) in the context of portfolio optimisation. Fully Flexible Views (Meucci, 2008) is a generalization of the previously developed models; BL model, the Copula Opinion Pooling model of Meucci (2006) and partial views on expectations and covariances by Piezer (2007). Similarly to the Black-Litterman model, the inputs of Entropy Pooling are a prior market distribution and a set of general views. The difference lays in the computation of the posterior distribution. When combining the views with the reference model, the posterior distribution should contain the least possible amount of spurious, fallacious structure; this is obtained by minimizing the entropy relative to the prior distribution. This can be achieved by only modifying the probabilities associated with the original historical scenarios or Monte Carlo simulations which remain unaltered.

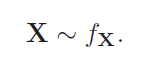
The table below showcases the advantages of the EP as compared to Black-Litterman (1990), Almgren and Chriss (2006), Qian and Gorman (2001), Pezier (2007), Meucci (2009) and the COP in Meucci (2006).

**A screenshot of a cell phone

Description automatically generated**

**The reference model**

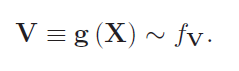
Similarly to the BL model, we begin with a reference model (“the prior”), determined through historical scenarios or Monte Carlo simulations, represented by its probability density function (pdf):

 (22)

where **X** is an N-dimensional vector of risk factors that can be but is by no means limited to a time series of returns on a set of securities.

**The views**

A practitioner can express views on generic functions of the market *g1* (X), …, *gK* (X), forming a K-dimensional random variable whose joint distribution is implied from the prior distribution (22).

 (23)

It is important to note that the functions *gk* don’t have to be linear, unlike in BL. The views are statements on the variables and are not necessarily in line with the reference model. In general, only certain features of the distribution of **V** are targeted by users’ views.

For instance, views can be added on the expected returns, and can be expressed as inequalities which are useful for ranking, statements on the volatilities, on correlations, on the tail behaviours, on the lower (upper) tail co-dependence among others. In Section 3.1, we will exhibit in more detail the conditioning applied in the algorithm used in this paper.

**The posterior**

The posterior distribution should be as close as possible to the reference model (22) while satisfying the views, without adding additional structure. The relative entropy between a reference distribution *f*X and a generic distribution ^*f*X

**A close up of a logo

Description automatically generated** (24)

is a measure of the amount of structure in ^*f*X; also indicating the distortion of the generic distribution ^*f*X with respect to the reference model *f*X. In the case when the two distributions coincide, the relative entropy equals to zero. The more constraints imposed by the user, the larger the distortion and therefore the higher the relative entropy.

The posterior market distribution is defined as

A close up of a logo

Description automatically generated (25)

where *f* ∈ V represents all the distributions consistent with the views expressed in (23).

**The confidence**

The posterior distribution ^*f*X assumes full confidence in the user’s views. If it is not the case, the posterior distribution ^ *f*X must tilt towards the reference distribution *f*X.

By opinion-pooling the full-confidence posterior and the reference model, as in Meucci (2006):

A close up of a clock

Description automatically generated (26)

where *c* ∈ [0, 1] is a pooling parameter and represents the confidence level in the views.

The Entropy pooling method also finds its use in a multi-manager context with *S* users incorporating their separate views. The views can target the same functions of the market but not necessarily. Having *S* different opinions leads to *S* full-confidence posterior distributions ^fX(s), *s* = 1, …, *S*. The combined posterior distribution is obtained by computing a confidence-weighted average of the *S* individual full-confidence posterior distributions:

A picture containing object, clock

Description automatically generated (27)

Meucci (2010) suggests that confidence could even be respectively linked to the managers’ track record. A manager having successful views or managers having a higher number of past views (i.e. seniority) would see the weight (confidence level) of their opinions increased.

* 1. **Performance Measures**

1. **Methodology**
   1. **Applying Fully Flexible Probabilities to Historical Scenarios**

“Fully Flexible Probabilities” (FFP) help portfolio managers to emphasise particular periods of historical scenarios where certain conditions are similar to the one the practitioner expects. The application of this model allows one to attribute different relative weights to each observation. We follow Meucci (2010) (2011b) to explain some illustrative implementations of FFP.

Consider *f*X as a *T* x *N* matrix of joint scenarios {xt,n} with a constant probability pt 1/T to occur. In the case risk drivers are independent and identically distributed random variables across time, the scenarios can be past realisations. In this paper, historical realisations will be used but Monte Carlo simulations could also be drawn.

In the following subsections, different relative weights will be assigned to historical scenarios through three kinds of conditioning: (1) time conditioning for which weights depend on the moment the scenario occurred; (2) state conditioning for which weights depend on market conditions at the time the scenario occurred and finally (3) joint time and state conditioning through the Minimum Relative Entropy approach. These same conditioning techniques appeared to give satisfactory results according to Denis and Scaillet (2017) and are therefore used in this paper.

It is relevant to note that historical scenarios are not altered by this approach. It is a matter of adjusting the relative weight of historical scenarios corresponding to the conditioning put in place by the practitioner. When necessary to re-normalise the probabilities, meaning to ensure that ΣTt=1 *p*t ≡ 1, the symbol ∝ will be used instead of ≡.

* + 1. **Time Conditioning**

Using time conditioning allows the risk or portfolio manager to focus on a precise period of time by associating a relative weight to a particular scenario depending on the moment it occurred.

* + - 1. **Time Crisp Conditioning**

The most straightforward form of conditioning is the rolling window. An arbitrary time window *T*t\* is chosen and the probability is constant over that period and zero otherwise:

A close up of text on a white background

Description automatically generated (28)

Assuming a portfolio manager would rely on more recent set of data, he will choose a window covering a fixed period *T*t\* from the last observation t\* ≡ T to t ≡ t\* - *T*t\*.

Cutting through historical scenarios in such an abrupt manner is typically avoided as observations will either receive a weight of 1/ *T*t\* or will not be considered at all. Generally, practitioners will favour time exponentially decaying probabilities providing smoother profile.

* + - 1. **Exponential Smoothing Conditioning**

The following conditioning can be applied to the probabilities {pt} in order to obtain smoother profile:

A close up of a sign

Description automatically generated (29)

where τ*HL* > 0 is the half-life representing the time needed for decaying probabilities to decrease to one half of their highest value in *T*. The lower the half-life τ*HL* the higher the decay rate and therefore observations less distant in time from *t\**, the target, will receive a lower weight.

* + 1. **State Conditioning**

State conditioning allows one to be able to emphasise scenarios similar to today’s expected market conditions. For instance, if the present market conditions show high inflation, the algorithm will give more weight to historical scenarios presenting equivalent conditions. Many macro-economic indicators could be used in this type of conditioning.

Consider Zt, a risk factor supposed independent and identically distributed across time and observable at each time t (daily in this study). We can collect a time series {wt}t=1T of historical scenarios. Through state conditioning we will stress scenarios where Wt falls in the range R(z\*) = [zd,zu] (the “emphasis zone”) around the target level w\*.

* + - 1. **State Crisp Conditioning**

The scenarios that occurred in the state window R(z\*) are fully weighted and the others have a probability of zero:

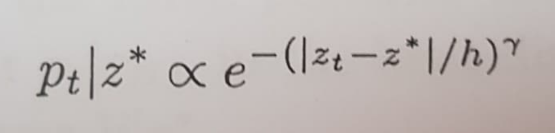
A close up of a logo

Description automatically generated (30)

The drawback is the same as in the Time Crisp Conditioning as it cuts too abruptly through data. An alternative would be to use a kernel in order to obtain smoother profile.

* + - 1. **Kernel Smoothing Conditioning**

Kernel Smoothing attributes weight to historical scenarios {*zt*} proportionally to their distance from the target *z\** as computed by a normal density function around the target:

 (31)

where *h* is the kernel bandwidth and γ ≡ 1 for exponential kernel probabilities, γ ≡ 2 for Gaussian kernel probabilities. The level of smoothing is defined by *h.* A large *h* gives probabilities closer to equal probabilities where *pt* ≈ 1/T whereas small *h* will yield *pt* tending to the crisp conditioning profile. This method is exhibited in the Figure 3.

* + 1. **Joint Time and State Conditioning with Entropy Minimisation**

Portfolio managers might be willing to emphasise a particular market state similar to recent conditions while stressing more recent historical scenarios to old ones. By applying Meucci (2008)’s MRE technique, it is possible to blend time and state conditioning. We describe this technique following Meucci (2011a) and Denis and Scaillet (2017).

First, exponentially decaying probabilities *pt* | τ*HL* as in (29) are set to define prior probabilities **p** ≡ (*p1*, …, *pt*). Then, views on expectations and standard deviation of the yet-to-be defined Flexible Probabilities **p** (*p1*, …, *pt*) are imposed:

A close up of text on a whiteboard

Description automatically generated (32)

where *m|z\** and *s2|z\** are respectively the “historical with Flexible Probabilities” mean and variance of the state conditioning variable *z\**:

A close up of a whiteboard

Description automatically generated (33)

A close up of text on a whiteboard

Description automatically generated (34)

in which *pt|*R(*z\**) are the State Crisp probabilities in (30).

The next step is to use the Entropy Pooling approach to combine the views **V**|*z\** with the prior *pt|* τ*HL*and to minimize the relative entropy:

A close up of a logo

Description automatically generated (35)

The resulting posterior probabilities *p|*(*z\**, τ*HL*) are a multiplication of the exponential smoothed prior (29) in addition to the multiplication of the best exponential kernel with the best Gaussian kernel:

A picture containing object, clock

Description automatically generated (36)

where the coefficients λ*m* , λ*s* are suitable Lagrange multipliers for the views respectively on the mean and on the variance in (32).

The results of the joint time and state conditioning using the MRE approach are illustrated in the Figure 3.

* 1. **Portfolio Optimisation**

The goal of optimising a portfolio is to maximise the satisfaction of the portfolio manager while respecting a set of constraints by finding optimal weights to each asset included in the portfolio.

A close up of a logo

Description automatically generated (37)

* + 1. **Two-Step Mean-Variance Approach**

According to Meucci (2005)[[4]](#footnote-4) (2011b), numerical optimisation is a challenging task and one can resort to use a two-step mean-variance heuristic. Using quadratic programming, a mean-variance efficient frontier is easily computed. This results in a one-parameter vector of efficient allocations from all the possible asset combinations.

A close up of a logo

Description automatically generated (38)

A simple univariate grid search allows then to find the optimal portfolio maximising the satisfaction of the practitioner.

A close up of a logo

Description automatically generated (39)

* + 1. **Global Minimum Variance Portfolio**

The portfolio that has the lowest variance on the mean-variance efficient frontier is the global minimum variance portfolio. To find that precise point on the frontier, we need to solve the following minimisation problem[[5]](#footnote-5):

A close up of a logo

Description automatically generated (40)

where *w* ≡ [*w*1, …, *w*N]’ is a column vector of portfolio weights, **1** is a column vector of appropriate dimension filled with ones and Σ is the covariance matrix of asset returns. The weights of the global minimum variance portfolio are given as

A picture containing object, clock

Description automatically generated (41)

In this paper, we assume that the manager’s satisfaction is simply defined by the opposite of the volatility of the returns distribution.

A close up of text on a whiteboard

Description automatically generated (42)

1. **Experimental Set-Up**
   1. **Data**

The data used for empirical experiments consists in a set of five different asset classes: Bonds, Stocks, Real Estate, Commodities and Cryptocurrency. The choice of assets is arbitrary, mostly chosen for their accessibility and liquidity to make this framework feasible for most portfolio managers.

The Fixed Income asset class is represented by the iShares Core U.S. Aggregate Bond ETF (AGG), picked because there is a low correlation with Equities. The Equity asset class is represented by the S&P500 as a large cap index, and by the Russell 2000 (IWM) for the small cap index. The Real Estate asset class is represented by the Vanguard Real Estate ETF (VNQ) as it has a broad, diversified exposure. The commodities asset class is represented by PowerShares DB Commodity Index Tracking Fund (DBC) allowing to track a basket of commodities, having the advantage to not be gold, silver, or oil specified. The crypto-currency asset class is represented by Bitcoin.

For this study we have gathered 6 years of daily data retrieved from Thomas Reuters and Yahoo Finance from June 1, 2013 to June 1, 2019 totalling 1260 date points. The following table provides descriptive statistics for each asset included in the portfolio. Concerning the state conditioning, two different macroeconomic indicators will be tested, namely the VIX and the Interest Rate on 10 year notes (^TNX), with data gathered over the same period of time.

* 1. **Implementation**

First, an asset allocation based on the buy-and-hold strategy will be applied to the portfolio. This passive strategy is the simplest form of portfolio management where the practitioner balances its portfolio at t=0 and holds its positions until the end of the investment horizon t=T. (Perold and Sharpe, 1988). This strategy entails no rebalancing for the whole period of investment, resulting in low transaction costs.

The prior market is consisted of the first two years of daily data from June 1, 2013 to June 1, 2015. Optimal asset weights are found using this sample. The remaining daily data from June 1, 2015 to June 1, 2019 is utilized to backtest the resulting asset allocation.

To add, a Walk-Forward Analysis (WFA) is carried out in order to perform a “rolling or periodic optimisation” (Pardo, 2008). This second backtesting framework allows to test the asset allocation on a set of multiple subsamples. First, it allows to get closer to a real trading simulation. Second, this method verifies the robustness of our portfolio. Third, the WFA framework allows to have higher performing portfolios due to its ability to capture changing market conditions thanks to its periodic re-optimisation (Peterson, 2015). With this second method, our conditioned portfolios follow the same buy-and-hold strategy but the prior market now consists in a rolling window of two years of data. The resulting allocation is tested for one month of “out-of-sample” data after which a new optimisation is performed. In other words, it presumes a monthly rebalancing while using the most recent two years daily data. Every month forward the first month of the in-sample is removed to welcome the more recent one month data. Our algorithm therefore performs 48 iterations.

* + 1. **Conditioning Setting**
    2. **Simulation Setting**

1. Results
   1. Performance Measures
   2. Test 1: Comparison with equally weighted, global minimum variance portfolio, Black-Litterman, minimum relative entropy
   3. Robustness tests
2. Conclusion

**Reference**

Markowitz, H. (1952). Portfolio selection. *The journal of finance*, *7*(1), 77-91.

Cheung, W. (2013). The augmented Black–Litterman model: A ranking-free approach to factor-based portfolio construction and beyond. *Quantitative Finance*, *13*(2), 301-316.

Harris, R. D., Stoja, E., & Tan, L. (2017). The dynamic Black–Litterman approach to asset allocation. *European Journal of Operational Research*, *259*(3), 1085-1096.

Meucci, A. (2010). Black–Litterman Approach. *Encyclopedia of Quantitative Finance*.

Martellini, L., & Ziemann, V. (2007). Extending Black-Litterman analysis beyond the mean-variance framework. *Journal of Portfolio Management*, *33*(4), 33.

Xiong, J. X., & Idzorek, T. M. (2011). The impact of skewness and fat tails on the asset allocation decision. *Financial Analysts Journal*, *67*(2), 23-35.

Meucci, A. (2006). Beyond Black-Litterman in practice: A five-step recipe to input views on non-normal markets. *Available at SSRN 872577*.

Meucci, A. (2005). Beyond Black-Litterman: Views on non-normal markets.

Denis, M., Scaillet, O., & Université libre de Bruxelles, degree granting institution. (2017). *Using Fully Flexible Probabilities for Strategic Asset Allocation : An Empirical Experiment on a Global Minimum Variance Portfolio*.

Jorion, P. (1985). International portfolio diversification with estimation risk. *Journal of Business*, 259-278.

Jorion, P. (2007). *Financial risk manager handbook* (Vol. 406). John Wiley & Sons.

Yanushevsky, R. (2015). An approach to improve mean-variance portfolio optimization model. *Journal of Asset Management*, *16*(3), 209-219.

Perold, A. F., & Sharpe, W. F. (1988). Dynamic strategies for asset allocation. *Financial Analysts Journal*, *44*(1), 16-27.

Peterson, B. G. (2015). Developing & Backtesting Systematic Trading Strate-gies. *DV Trading. http://goo. gl/na4u5d*.

DeMiguel, V., Garlappi, L., & Uppal, R. (2007). Optimal versus naive diversification: How inefficient is the 1/N portfolio strategy?. *The review of Financial studies*, *22*(5), 1915-1953.

Merton, R. C. (1980). On estimating the expected return on the market: An exploratory investigation. *Journal of financial economics*, *8*(4), 323-361.

Chopra, V. K., & Ziemba, W. T. (2013). The effect of errors in means, variances, and covariances on optimal portfolio choice. In *Handbook of the Fundamentals of Financial Decision Making: Part I* (pp. 365-373).

Ceria, S., & Stubbs, R. A. (2006). Incorporating estimation errors into portfolio selection: Robust portfolio construction. *Journal of Asset Management*, *7*(2), 109-127.

Goldfarb, D., & Iyengar, G. (2003). Robust portfolio selection problems. *Mathematics of operations research*, *28*(1), 1-38.

Bevan, A., & Winkelmann, K. (1998). Using the Black-Litterman Global Asset Allocation Model. *Goldman Sachs*.

Bertsimas, D., Gupta, V., & Paschalidis, I. C. (2012). Inverse optimization: A new perspective on the Black-Litterman model. *Operations research*, *60*(6), 1389-1403.

Idzorek, T. (2007). A step-by-step guide to the Black-Litterman model: Incorporating user-specified confidence levels. In *Forecasting expected returns in the financial markets* (pp. 17-38). Academic Press.

Best, M. J., & Grauer, R. R. (1991). On the sensitivity of mean-variance-efficient portfolios to changes in asset means: some analytical and computational results. *The review of financial studies*, *4*(2), 315-342.

Black, F., & Litterman, R. (1992). Global portfolio optimization. *Financial analysts journal*, *48*(5), 28-43.

Cheung, W. (2010). The black–litterman model explained. *Journal of Asset Management*, *11*(4), 229-243.

Peiro, A. (1999). Skewness in financial returns. *Journal of Banking & Finance*, *23*(6), 847-862.

Ang, A., & Chen, J. (2002). Asymmetric correlations of equity portfolios. *Journal of financial Economics*, *63*(3), 443-494.

Bollerslev, T., Engle, R. F., & Wooldridge, J. M. (1988). A capital asset pricing model with time-varying covariances. *Journal of political Economy*, *96*(1), 116-131.

Scott, R. C., & Horvath, P. A. (1980). On the direction of preference for moments of higher order than the variance. *The Journal of Finance*, *35*(4), 915-919.

Andersen, T. G., Bollerslev, T., & Diebold, F. X. (2010). Parametric and nonparametric volatility measurement. In *Handbook of Financial Econometrics: Tools and Techniques* (pp. 67-137). North-Holland.

Meucci, A. (2011a). Mixing Probabilities, Priors and Kernels via Entropy Pooling. *GARP Risk Professional*, 32-36.

Meucci, A. (2011b). ‘The Prayer’Ten-Step Checklist for Advanced Risk and Portfolio Management. *Ten-Step Checklist for Advanced Risk and Portfolio Management (February 2, 2011)*.

1. Historical scenarios enhancement thourhg Fully Flexible Probabilities is a concept described by Meucci (2010) for risk management purposes. In his paper, he explains how changing the probability distribution can be used to “reflect specific market conditions, advanced estimation techniques [and] partial information, using the entropy-based technique”. In short, our original contribution comes from our application of this theory to the optimisation of a multi-asset portfolio. [↑](#footnote-ref-1)
2. We refer to these assets as different types of assets as defined in CFA Program Curriculum, Level 1, Volume 6. [↑](#footnote-ref-2)
3. Source : Idzorek, T. (2007). A step-by-step guide to the Black-Litterman model: Incorporating user-specified confidence levels. In *Forecasting expected returns in the financial markets* (pp. 17-38). Academic Press. [↑](#footnote-ref-3)
4. In his book Risk and Asset Allocation (2005), in chapter 6 Meucci discusses the various ways to optimize allocations, among others the two-step mean-variance framework is fully detailed and explained pg 301-357. [↑](#footnote-ref-4)
5. See Bodnar et al. (2017) for more information. [↑](#footnote-ref-5)